

On Sobolev's mappings on Riemann surfaces

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Abstract

In terms of dilatations, it is proved a series of criteria for continuous and homeomorphic extension to the boundary of mappings with finite distortion between regular domains on the Riemann surfaces

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1 Introduction

Recall that n –**dimensional topological manifold** \mathbb{M}^n is a Hausdorff topological space with a countable base every point of which has an open neighborhood that is homeomorphic to \mathbb{R}^n or, the same, to an open ball in \mathbb{R}^n , see e.g. [5]. A **chart on the manifold** \mathbb{M}^n is a pair (U, g) where U is an open subset of the space \mathbb{M}^n and g is a homeomorphism of U on an open subset of the coordinate space \mathbb{R}^n . Note that \mathbb{R}^2 is homeomorphic to \mathbb{C} through the correspondence $(x, y) \Rightarrow z := x + iy$.

A **complex chart** on the two-dimensional manifold \mathbb{S} is a homeomorphism g of an open set $U \subseteq \mathbb{S}$ onto an open set $V \subseteq \mathbb{C}$ under that every point $p \in U$ corresponds a number z , its **local coordinate**. The set U itself is sometimes called a chart. Two complex charts $g_1 : U_1 \rightarrow V_1$ and $g_2 : U_2 \rightarrow V_2$ are called **conformal confirmed** if the map

$$g_2 \circ g_1^{-1} : g_1(U_1 \cap U_2) \rightarrow g_2(U_1 \cap U_2) \quad (1.1)$$

is conformal. A **complex atlas** on \mathbb{S} is a collection of mutually conformal confirmed charts covering \mathbb{S} . Complex atlases on \mathbb{S} are called conformal confirmed if their charts are so.

A **complex structure** on a two-dimensional manifold \mathbb{S} is an equivalence class of conformal confirmed atlases on \mathbb{S} . It is clear that a complex structure on \mathbb{S} can be determined by one of its atlases. Moreover, uniting all atlases of a complex structure on \mathbb{S} , we obtain its atlas Σ that is maximal by inclusion. Thus, the complex structure can be identified with its maximal atlas Σ . The **conjugate complex structure** $\bar{\Sigma}$ on \mathbb{S} consists of the charts \bar{g} of the complex conjugation of $g \in \Sigma$ that connected each to other by the anti-conformal mapping of \mathbb{C} of the mirror reflection with respect to the real axis not keeping orientation. Thus, we have no uniqueness for the complex structures on two-dimensional manifolds.

A **Riemann surface** is a pair (\mathbb{S}, Σ) consisting of a two-dimensional manifold \mathbb{S} and a complex structure Σ on \mathbb{S} . As usual, it is written only \mathbb{S} instead of (\mathbb{S}, Σ) if the choice of the complex structure Σ is clear by a context. Given a Riemann surface \mathbb{S} , a **chart** on \mathbb{S} is a complex chart in the maximal atlas of its complex structure.

Now, let \mathbb{S} and \mathbb{S}^* be Riemann surfaces. We say that a mapping $f : \mathbb{S} \rightarrow \mathbb{S}^*$ belongs to the Sobolev class $W_{\text{loc}}^{1,1}$ if f belongs to $W_{\text{loc}}^{1,1}$ in local coordinates, i.e., if for every point $p \in \mathbb{S}$ there exist charts $g : U \rightarrow V$ and $g_* : U_* \rightarrow V_*$ on \mathbb{S} and \mathbb{S}^* , correspondingly, such that $p \in U$, $f(U) \subseteq U_*$ and the mapping

$$F := g_* \circ f \circ g^{-1} : V \rightarrow V_* \quad (1.2)$$

belongs to the class $W_{\text{loc}}^{1,1}$. Note that the latter property is invariant under replacements of charts because the class $W_{\text{loc}}^{1,1}$ is invariant with respect to replacements of variables in \mathbb{C} that are local quasiisometries, see e.g. Theorem 1.1.7 in [21], and conformal mappings are so in view of boundedness of their derivatives on compact sets. Note also that domains D and D^* , i.e. open connected sets, on Riemann surfaces \mathbb{S} and \mathbb{S}^* are themselves Riemann surfaces with complex structures induced by the complex structures on \mathbb{S} and \mathbb{S}^* , correspondingly. Hence the definition given above can be extended to mappings $f : D \rightarrow D_*$.

Recall also that functions of the class $W_{\text{loc}}^{1,1}$ in \mathbb{C} are absolutely continuous on lines, see e.g. Theorem 1.1.3 in [21], and, consequently, almost everywhere have partial derivatives. By the Gehring-Lehto theorem such complex-valued

functions also have almost everywhere the total differential if they are open mappings, i.e., if they map open sets onto open sets, see [8]. Note that this result was before it obtained by Menshov for homeomorphisms and, moreover, his proof can be extended to open mappings with no changes, see [22]. We will apply this fact just to homeomorphisms. It is clear that the property of differentiability of mappings at a point is invariant with respect to replacements of local coordinates on Riemann surfaces. Note that, under the research of the boundary behavior of homeomorphisms f between domains on Riemann surfaces, it is sufficient to be restricted by sense preserving homeomorphisms because in the case of need we may pass to the conjugate complex structure in the image.

2 Definitions and preliminary remarks

First of all note that by the Uryson theorem topological manifolds are metrizable because they are Hausdorff regular topological spaces with a countable base, see [29] or Theorem 22.II.1 in [17].

As well-known, see e.g. Section III.III.2 in [28], the Riemann surfaces are orientable two-dimensional manifolds and, inversely, orientable two-dimensional manifolds admit complex structures, i.e., are supports of Riemann surfaces, see e.g. Section III.III.3 in [28], see also Theorem 6.1.9 in [31]. Moreover, two-dimensional topological manifolds are triangulable, see e.g. Section III.II.4 in [28], see also Theorem 6.1.8 in [31].

Every orientable two-dimensional manifold \mathbb{S} has the **canonical representation of Kerekjarto-Stoilow** in the form of a part of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that appears after removing from \mathbb{C} a compact totally disconnected set B of points of the real axis and of a finite or countable collection of pairs mutually disjoint disks that are symmetric with respect to the real axis whose boundary circles can be accumulated only to the set B and whose points pairwise identified, see e.g. III.III in [28]. The number g of these pairs of glued circles is called a **genus of the surface** \mathbb{S} .

It is clear that the topological model of Kerekjarto-Stoilow is homeomorphic to the sphere $\mathbb{S}^2 \simeq \overline{\mathbb{C}}$ in \mathbb{R}^3 with g handles and a compact totally disconnected set of punctures in \mathbb{S}^2 . Gluing these punctures in the Kerekjarto-Stoilow model by points of the set B , we obtain a compact topological space that is not a two-dimensional manifold if $g = \infty$. Similarly, joining the boundary elements to the initial surface \mathbb{S} , that correspond in a one-to-one manner to the points of the set B , we obtain its **compactification by Kerekjarto-Stoilow** $\overline{\mathbb{S}}$.

Next, let x_k , $k = 1, 2, \dots$, be a sequence of points in a topological space X . It is said that a point $x_* \in X$ is a **limit point** of the sequence x_k , written $x_* = \lim_{k \rightarrow \infty} x_k$ or simply $x_k \rightarrow x_*$ if every neighborhood U of the point x_* contains all points of the sequence except its finite collection. Let Ω and Ω_* be open sets in topological spaces X and X_* , correspondingly. Later on, $C(x, f)$ denotes the **cluster set** of a mapping $f : \Omega \rightarrow \Omega_*$ at a point $x \in \overline{\Omega}$, i.e.,

$$C(x, f) := \left\{ x_* \in X_* : x_* = \lim_{k \rightarrow \infty} f(x_k), x_k \rightarrow x, x_k \in \Omega \right\} \quad (2.1)$$

It is known that the inclusion $C(x, f) \subseteq \partial\Omega_*$, $x \in \partial\Omega$, holds for homeomorphisms $f : \Omega \rightarrow \Omega_*$ in metric spaces, see e.g. Proposition 2.5 in [23] or Proposition 13.5 in [20]. Hence we have the following conclusion.

Proposition 2.1 *Let Ω and Ω_* be open sets on manifolds \mathbb{M}^n and \mathbb{M}_*^n , correspondingly, and let $f : \Omega \rightarrow \Omega_*$ be a homeomorphism. Then*

$$C(p, f) \subseteq \partial\Omega_* \quad \forall p \in \partial\Omega \quad (2.2)$$

In particular, we come from here to the following statement.

Corollary 2.1 *Let D and D_* be domains on Riemann surfaces \mathbb{S} and \mathbb{S}_* , correspondingly, and let $f : D \rightarrow D_*$ be a homeomorphism. Then*

$$C(\partial D, f) := \bigcup_{p \in \partial D} C(p, f) \subseteq \partial D_* \quad (2.3)$$

Now, let us give the main result of the theory of uniformization of Riemann surfaces that will be essentially applied later on, see e.g. Section II.3 in [16]. The **Poincare uniformization theorem** (1908) states that every Riemann

surface \mathbb{S} is represented (up to the conformal equivalence) in the form of the factor $\tilde{\mathbb{S}}/G$ where $\tilde{\mathbb{S}}$ is one of the canonical domains: $\overline{\mathbb{C}}$, \mathbb{C} or the unit disk \mathbb{D} in \mathbb{C} and G is a discrete group of conformal (= fractional) mappings of $\tilde{\mathbb{S}}$ onto itself. The corresponding Riemann surfaces are called of **elliptic, parabolic and hyperbolic type**.

Moreover, $\tilde{\mathbb{S}} = \overline{\mathbb{C}}$ only in the case when \mathbb{S} is itself conformally equivalent to the sphere $\overline{\mathbb{C}}$ and the group G is trivial, i.e., consists only of the identity mapping; $\tilde{\mathbb{S}} = \mathbb{C}$ if \mathbb{S} is conformally equivalent to either \mathbb{C} , $\mathbb{C} \setminus \{0\}$ or a torus and, correspondingly, the group G is either trivial or is a group of shifts with one generator $z \rightarrow z + \omega$, $\omega \in \mathbb{C} \setminus \{0\}$ or a group of shifts with two generators $z \rightarrow z + \omega_1$ and $z \rightarrow z + \omega_2$ where ω_1 and $\omega_2 \in \mathbb{C} \setminus \{0\}$ and $\text{Im } \omega_1/\omega_2 > 0$. Except these simplest cases, every Riemann surface \mathbb{S} is conformally equivalent to the unit disk \mathbb{D} factored by a discrete group G without fixed points, see e.g. Theorem 7.4.2 in [31]. And inversely, every factor \mathbb{D}/G is a Riemann surface, see e.g. Theorem 6.2.1 [2].

In this connection, recall that we identify in the factor $\tilde{\mathbb{S}}/G$ all elements of the **orbit** $G_{z_0} := \{ z \in \tilde{\mathbb{S}} : z = g(z_0), g \in G \}$ of every point $z_0 \in \tilde{\mathbb{S}}$. Recall also that a group G of fractional mappings of \mathbb{D} onto itself is called **discrete** if the unit of G (the identical mapping I) is an isolated element of G . As easy to see, the latter implies that all elements of the group G are isolated each to other. If the elements of the group G have no fixed points as in the uniformization theorem, then the latter is equivalent to that the group G **discontinuously acts** on \mathbb{D} , i.e., for every point $z \in \mathbb{D}$, there is its neighborhood U such that $g(U) \cap U = \emptyset$ for all $g \in G$, $g \neq I$, see e.g. Theorem 8.4.1 in [2].

Let us also describe in short the **Poincare model** of non-Euclidean plane, in other words, the so-called Boyai-Gauss-Lobachevskii geometry or the hyperbolic geometry. Points of the hyperbolic plane are points of the unit disk \mathbb{D} and **hyperbolic straight lines** are the arcs in \mathbb{D} of circles that are perpendicular to the unit circle $\mathbb{S}^1 := \partial\mathbb{D}$ and the diameters of \mathbb{D} . Every two points in \mathbb{D} determine exactly a single hyperbolic straight line, see e.g. Proposition 7.2.2 in

[31]. The **hyperbolic distance** in the unit disk \mathbb{D} is given by the formula

$$h(z_1, z_2) = \log \frac{1+t}{1-t}, \quad \text{where} \quad t = \frac{|z_1 - z_2|}{|1 - z_1 \bar{z}_2|}, \quad (2.4)$$

the **hyperbolic length** of a curve γ and the **hyperbolic area** of a set S in \mathbb{D} are calculated as the integrals, see e.g. Section 7.1 in [2], Proposition 7.2.9 in [16],

$$s_h(\gamma) = \int_{\gamma} \frac{2|dz|}{1-|z|^2}, \quad h(S) = \int_S \frac{4dx dy}{(1-|z|^2)^2}, \quad \text{where} \quad z = x + iy. \quad (2.5)$$

All conformal (= fractional) mappings of \mathbb{D} onto itself are **hyperbolic isometries**, i.e., they keep the hyperbolic distance, see e.g. Theorem 7.4.1 in [2], and hence the hyperbolic length as well as the hyperbolic area are invariant under such mappings.

A **hyperbolic half-plane** H , i.e., one of two connected components of the complement of a hyperbolic straight line L in \mathbb{D} , is a **hyperbolically convex set**, i.e., every two points in H can be connected by a segment of a hyperbolic straight line in H , see e.g. [2], p. 128. A **hyperbolic polygon** is a domain in \mathbb{D} bounded by a Jordan curve, consisting of segments of hyperbolic straight lines. If G is a discrete group of fractional mappings of \mathbb{D} onto itself without fixed points, then the **Dirichlet polygon** for G with the center $\zeta \in \mathbb{D}$ is the convex set

$$D_{\zeta} = \bigcap_{g \in G, g \neq I} H_g(\zeta) \quad (2.6)$$

where

$$H_g(\zeta) = \{z \in \mathbb{D} : h(z, \zeta) < h(z, g(\zeta))\}$$

is a hyperbolic half-plane containing the point ζ and bounded by the hyperbolic straight line $L_g(\zeta) = \{z \in \mathbb{D} : h(z, \zeta) = h(z, g(\zeta))\}$. D_{ζ} is also called the **Poincare polygon**. Dirichlet applied this construction at 1850 for the Euclidean spaces and, later on, Poincare has applied it to hyperbolic spaces.

The geometric approach to the study of the factors \mathbb{D}/G is based on the notion of its fundamental domains. A **fundamental set** for the group G is a set F in \mathbb{D} containing precisely one point z in every orbit G_{z_0} , $z_0 \in \mathbb{D}$. Thus,

$\bigcup_{g \in G} g(F) = \mathbb{D}$. The existence of a fundamental set is guaranteed by the choice axiom, see e.g. [30], p. 246. A domain $D \subset \mathbb{D}$ is called a **fundamental domain** for G if there is a fundamental set F for G such that $D \subset F \subset \overline{D}$ and $h(\partial D) = 0$. If D is a fundamental domain for a discrete group G of fractional mappings \mathbb{D} onto itself without fixed points, then D and its images pave \mathbb{D} , i.e.,

$$\bigcup_{g \in G} g(\overline{D}) = \mathbb{D}, \quad g(D) \cap D = \emptyset \quad \forall g \in G, g \neq I. \quad (2.7)$$

The Poincare polygon is an example of a fundamental domain that there is for every such a group, see e.g. Theorem 9.4.2 in [2].

The **hyperbolic distance on a factor** \mathbb{D}/G for a discrete group G without fixed points can be defined in the following way. Let p_1 and $p_2 \in \mathbb{D}/G$. Then by the definition p_1 and p_2 are orbits G_{z_1} and G_{z_2} of points z_1 and $z_2 \in \mathbb{D}$. Set

$$h(p_1, p_2) = \inf_{g_1, g_2 \in G} h(g_1(z_1), g_2(z_2)). \quad (2.8)$$

In view of discontinuous action of the group G , no orbit have limit points inside of \mathbb{D} and, by the invariance of hyperbolic metric in \mathbb{D} with respect to the group G , we have

$$\begin{aligned} h(p_1, p_2) &= \min_{g_1, g_2 \in G} h(g_1(z_1), g_2(z_2)) = \\ &= \min_{g \in G} h(z_1, g(z_2)) = \min_{g \in G} h(g(z_1), z_2). \end{aligned} \quad (2.9)$$

It is easy to see from here that $h(p_1, p_2) = h(p_2, p_1)$ and that $h(p_1, p_2) \neq 0$ if $p_1 \neq p_2$. It remains to show the triangle inequality. Indeed, let $p_0 = G_{z_0}$, $p_1 = G_{z_1}$ and $p_2 = G_{z_2}$ and let $h(p_0, p_1) = h(z_0, g_1(z_1))$ and $h(p_0, p_2) = h(z_0, g_2(z_2))$. Then we conclude from (2.9) that

$$h(p_1, p_2) \leq h(g_1(z_1), g_2(z_2)) \leq h(z_0, g_1(z_1)) + h(z_0, g_2(z_2)) = h(p_0, p_1) + h(p_0, p_2).$$

Now, let $\pi : \mathbb{D} \rightarrow \mathbb{D}/G$ be the natural projection and let F be a fundamental set in \mathbb{D} for the group G . Let us consider in F the metric

$$d(z_1, z_2) := h(\pi(z_1), \pi(z_2)). \quad (2.10)$$

Note that by the construction $d(z_1, z_2) \leq h(z_1, z_2)$ and, furthermore, $d(z_1, z_2) = h(z_1, z_2)$ if z_2 is close enough to z_1 in the hyperbolic metric in \mathbb{D} . Thus, we

obtain a metric space (F, d) that is homeomorphic to \mathbb{D}/G where the length and the area are calculated by the same formulas (2.5). Note that the elements of the length and the area in the integrals (2.5)

$$ds_h(z) = \frac{2|dz|}{1-|z|^2}, \quad dh(z) = \frac{4dx dy}{(1-|z|^2)^2}, \quad \text{где } z = x + iy, \quad (2.11)$$

are invariant with respect to fractional mappings of \mathbb{D} onto itself, i.e., they are functions of the point $p \in \mathbb{D}/G$ and hence they make possible to calculate the length and the area on the Riemann surfaces \mathbb{D}/G with no respect to the choice of the fundamental set F and the corresponding local coordinates.

For visuality, later on we sometimes identify \mathbb{D}/G with a fundamental set F in \mathbb{D} for the group G containing a fundamental (Dirichlet-Poincare) domain for G . The factor \mathbb{D}/G has a natural complex structure for which the projection $\pi : \mathbb{D} \rightarrow \mathbb{D}/G$ is a holomorphic (single-valued analytic) function whose restriction to every fundamental domain is a conformal mapping and, consequently, its inverse mapping is a complex chart of the Riemann surface \mathbb{D}/G .

It is clear that the distance (2.8), the elements of length and area (2.11) do not depend on the choice of G in the Poincare uniformization theorem because they are invariant under fractional mappings of \mathbb{D} onto itself and we call them **hyperbolic** on the Riemann surface \mathbb{S} .

The case of a torus \mathbb{S} is similar and much more simple, and hence it is not separately discussed. In this case, we set $s_h(z) = |dz|$ and $dh = dx dy$ but without the given name. The latter elements of length and area are also invariant under the corresponding (complex) proportional shifts in the Poincare uniformization theorem but up to the corresponding multiplicative constants.

Given a family Γ of paths γ in \mathbb{S} , a Borel function $\varrho : \mathbb{S} \rightarrow [0, \infty]$ is called **admissible** for Γ , abbr. $\varrho \in \text{adm } \Gamma$, if

$$\int_{\gamma} \varrho(p) ds_h(p) \geq 1 \quad (2.12)$$

for all $\gamma \in \Gamma$. The **modulus** of Γ is given by the equality

$$M(\Gamma) = \inf_{\varrho \in \text{adm } \Gamma} \int_{\mathbb{S}} \varrho^2(p) dh(p). \quad (2.13)$$

3 On mappings with finite distortion, the main lemma.

Recall that a homeomorphism f between domains D and D^* in \mathbb{R}^n , $n \geq 2$, is called of **finite distortion** if $f \in W_{\text{loc}}^{1,1}$ and

$$\|f'(x)\|^n \leq K(x) \cdot J_f(x) \quad (3.1)$$

with a function K that is a.e. finite. As usual, here $f'(x)$ denotes the Jacobian matrix of f at $x \in D$ where it is determined, $J_f(x) = \det f'(x)$ is the Jacobian of f at x , and $\|f'(x)\|$ is the operator norm of $f'(x)$, i.e.,

$$\|f'(x)\| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}. \quad (3.2)$$

First this notion was introduced in the plane for $f \in W_{\text{loc}}^{1,2}$ in the paper [10]. Later on, this condition was replaced by $f \in W_{\text{loc}}^{1,1}$, however, with the additional request $J_f \in L_{\text{loc}}^1$, see [11]. Note that the latter request can be omitted for homeomorphisms. Indeed, for every homeomorphism f between domains D and D^* in \mathbb{R}^n with first partial derivatives a.e. in D , there is a set E of the Lebesgue measure zero such that f has (N) -property of Lusin on $D \setminus E$ and

$$\int_A J_f(x) \, dm(x) = |f(A)| \quad (3.3)$$

for every Borel set $A \subset D \setminus E$, see e.g. 3.1.4, 3.1.8 and 3.2.5 in [4].

In the complex plane, $\|f'\| = |f_z| + |f_{\bar{z}}|$ and $J_f = |f_z|^2 - |f_{\bar{z}}|^2$ where

$$f_{\bar{z}} = (f_x + if_y)/2, \quad f_z = (f_x - if_y)/2, \quad z = x + iy,$$

and f_x and f_y are partial derivatives of f in x and y , correspondingly. Thus, in the case of sense-preserving homeomorphisms $f \in W_{\text{loc}}^{1,1}$, (3.1) is equivalent to the condition that $K_f(z) < \infty$ a.e. where

$$K_f(z) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \quad (3.4)$$

if $|f_z| \neq |f_{\bar{z}}|$, 1 if $f_z = 0 = f_{\bar{z}}$, and ∞ in the rest cases. As usual, the quantity $K_f(z)$ is called **dilatation** of the mapping f at z .

If $f : D \rightarrow D^*$ is a homeomorphism of the class $W_{\text{loc}}^{1,1}$ between domains D and D^* on the Riemann surfaces \mathbb{S} and \mathbb{S}^* , then $K_f(z)$ denotes the dilatation of the

mapping f in local coordinates, i.e., the dilatation of the mapping F in (1.2). The geometric sense of the quantity (3.4) at a point z of differentiability of the mapping f is the ratio of half-axes of the infinitesimal ellipse into which the infinitesimal circle centered at the point is transferred under the mapping f . The given quantity is invariant under the replacement of local coordinates, because conformal mappings transfer infinitesimal circles into infinitesimal circles and infinitesimal ellipses into infinitesimal ellipses with the same ratio of half-axes, i.e., K_f is really a function of a point $p \in \mathbb{S}$ but not of local coordinates.

We will call a homeomorphism $f : D \rightarrow D^*$ between domains D and D^* on Riemann surfaces \mathbb{S} and \mathbb{S}^* by a **mapping with finite distortion** if f is so in local coordinates. It is clear that this property enough to verify only for one atlas because conformal mappings have (N) –property of Lusin. We will say also that a homeomorphism $f : D \rightarrow D^*$ between domains D and D^* in the compactifications of Kerekjarto-Stoilow $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}^*}$ is a mapping with finite distortion if this property holds for its restriction to \mathbb{S} . Note that a homeomorphism between domains in \mathbb{S} and \mathbb{S}^* is always extended to a homeomorphisms between the corresponding domains in $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}^*}$. Later on, we assume that K_f is extended by zero outside of D and write $K_f \in L^1_{\text{loc}}$ if K_f is locally integrable with respect to the area h on \mathbb{S} .

Lemma 3.1 *Let D and D^* be domains on Riemann surfaces \mathbb{S} and \mathbb{S}^* . If $f : D \rightarrow D^*$ is a homeomorphism of finite distortion with $K_f \in L^1_{\text{loc}}$, then*

$$M(\Delta(fC_1, fC_2; fA)) \leq \int_A K_f(p) \cdot \xi^2(h(p, p_0)) dh(p) \quad \forall p_0 \in \overline{D} \quad (3.5)$$

for every ring $A = A(p_0, R_1, R_2) = \{p \in \mathbb{S} : R_1 < h(p, p_0) < R_2\}$, the circles $C_1 = \{p \in \mathbb{S} : h(p, p_0) = r_1\}$, $C_2 = \{p \in \mathbb{S} : h(p, p_0) = r_2\}$, $0 < R_1 < R_2 < \varepsilon = \varepsilon(p_0)$, and every measurable function $\xi : (R_1, R_2) \rightarrow [0, \infty]$ such that

$$\int_{R_1}^{R_2} \xi(R) dR \geq 1. \quad (3.6)$$

Proof. As it was discussed in Section 2, here we identify the Riemann surface \mathbb{D}/G with a fundamental set F in \mathbb{D} for G with the metric d defined by (2.10)

that contains a fundamental polygon of Poincare D_{z_0} for G centered at a point $z_0 \in \mathbb{D}$ whose orbit G_{z_0} is p_0 . With no loss of generality we may assume that $z_0 = 0$. The latter always can be obtained with the help of the fractional mapping of \mathbb{D} onto itself $g_0(z) = (z - z_0)/(1 - z\overline{z_0})$ transferring the point z_0 into the origin. Passing to the new group G_0 we obtain the Riemann surface \mathbb{D}/G_0 that is conformally equivalent to \mathbb{D}/G . Set

$$\delta_0 = \min \left[\inf_{\zeta \in \partial D_0} d(0, \zeta), \sup_{z \in D} d(0, z) \right] .$$

Let us choose $\delta \in (0, \delta_0)$ so small that, for $d(0, z) \leq \delta$, the equality $d(0, z) = h(0, z)$ holds. Note that correspondingly to (2.4)

$$R := h(0, z) = \log \frac{1+r}{1-r} , \quad \text{where} \quad r := |z| ,$$

and, correspondingly,

$$dR = \frac{2dr}{1-r^2} , \quad r = \frac{e^R - 1}{e^R + 1} .$$

Consequently,

$$\int_{r_1}^{r_2} \eta(r) dr \geq 1$$

where

$$\eta(r) = \frac{2}{1-r^2} \cdot \xi \left(\log \frac{1+r}{1-r} \right)$$

and, moreover,

$$\int_A K_f(z) \cdot \xi^2(d(z, z_0)) dh(z) = \int_A K_f(z) \cdot \eta^2(|z|) dm(z) \quad (3.7)$$

where the element of the area $dm(z) := dx dy$ corresponds to the Lebesgue measure in the plane \mathbb{C} . Moreover, note that $A = \{z \in \mathbb{D} : r_1 < |z| < r_2\}$, $C_1 = \{z \in \mathbb{D} : |z| = r_1\}$ и $C_2 = \{z \in \mathbb{D} : |z| = r_2\}$.

It is clear that the subset of the complex plane $D(\delta) := \{z \in D : |z| < \delta\}$ is decomposed into at most a countable collection of domains. Then components of the set $f(D(\delta))$ are homeomorphic to these domains and, consequently, by the general principle of Koebe, see e.g. Section II.3 in [16], they are conformally equivalent to plane domains, i.e., the family of curves $\Delta(fC_1, fC_2; fA)$

is decomposed into a countable collection of its subfamilies, belonging to the corresponding mutually disjoint complex charts of the Riemann surface \mathbb{D}/G^* . Thus, the conclusion of our lemma follows from Theorem 3 in [13]. \square

Remark 3.1 In other words, the statement of Lemma 3.1 means that every homeomorphism f of finite distortion between domains on Riemann surfaces with $K_f \in L^1_{\text{loc}}$ is the so-called ring Q –homeomorphism with $Q = K_f$. Note also that Riemann surfaces are locally the so-called Ahlfors 2–regular spaces with the mentioned metric and measure h , see e.g. Theorem 7.2.2 in [2]. Hence further we may apply results of the paper [27] on the boundary behavior of ring Q –homeomorphisms in metric spaces to homeomorphisms with finite distortion between domains on Riemann surfaces. It makes possible us, in comparison with the papers [25] and [26], to formulate new results in terms of the metric and measure h but not in terms of local coordinates on Riemann surfaces. Recall that the boundary behavior of Sobolev's homeomorphisms on smooth Riemannian manifolds for $n \geq 3$ was investigated in the paper [1].

4 On weakly flat and strongly accessible boundaries

In this section, we follow paper [23], see also Chapter 13 in monograph [20].

Later on, given sets E, F and Ω in a Riemann surface \mathbb{S} , $\Delta(E, F; \Omega)$ denotes the family of all curves $\gamma : [a, b] \rightarrow \mathbb{S}$ that join the sets E and F in Ω , i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in \Omega$ for $a < t < b$.

It is said that the boundary of a domain D in \mathbb{S} is **weakly flat at a point** $z_0 \in \partial D$ if, for every neighborhood U of the point z_0 and every number $N > 0$, there is a neighborhood $V \subset U$ of the point z_0 such that

$$M(\Delta(E, F; D)) \geq N \tag{4.1}$$

for all continua E and F in D intersecting ∂U and ∂V . The boundary of D is called **weakly flat** if it is weakly flat at every point in ∂D . Note that smooth and Lipschitz boundaries are weakly flat.

It is also said that a point $z_0 \in \partial D$ is **strongly accessible** if, for every neighborhood U of the point z_0 there exist a continuum E in D , a neighborhood $V \subset U$ of the point z_0 and a number $\delta > 0$ such that

$$M(\Delta(E, F; D)) \geq \delta \quad (4.2)$$

for every continuum F in D intersecting ∂U and ∂V . The boundary of D is called **strongly accessible** if every point $z_0 \in \partial D$ is so.

It is easy to see that if the boundary of a domain D in \mathbb{S} is weakly flat at a point $z_0 \in \partial D$, then the point z_0 is strongly accessible from D . Moreover, it was proved in metric spaces with measures that if a domain D is weakly flat at a point $z_0 \in \partial D$, then D is locally connected at z_0 , see e.g. Lemma 3.1 in [23] or Lemma 13.1 in [20].

Proposition 4.1 *If a domain D on a Riemann surface \mathbb{S} is weakly flat at a point in ∂D , then D is locally connected at the point.*

Recall that a domain D is called **locally connected at a point** in ∂D if, for every neighborhood U of the point, there is its neighborhood $V \subseteq U$ such that $V \cap D$ is a domain.

5 On extending to the boundary of the inverse mappings

In contrast with the direct mappings, see the next section, we have the following simple criterion for the inverse mappings.

Theorem 5.1 *Let \mathbb{S} and \mathbb{S}^* be Riemann surfaces, D and D^* be domains in \mathbb{S} and \mathbb{S}^* , correspondingly, $\partial D \subset \mathbb{S}$ and $\partial D^* \subset \mathbb{S}^*$, D be locally connected on its boundary and let ∂D^* be weakly flat. Suppose that $f : D \rightarrow D^*$ is a homeomorphism of finite distortion with $K_f \in L^1_{\text{loc}}$. Then the inverse mapping $g = f^{-1} : D^* \rightarrow D$ can be extended by continuity to a mapping $g : \overline{D^*} \rightarrow \overline{D}$.*

As it was before, we assume here that the dilatation K_f is extended by zero outside of the domain D .

Proof. By the Uryson theorem, see e.g. Theorem 22.II.1 in [17], $\bar{\mathbb{S}}$ is a metrizable space. Hence the compactness of $\bar{\mathbb{S}}$ is equivalent to its sequential compactness, see e.g. Remark 41.I.3 in [18], and the closure \bar{D} is a compact subset of \mathbb{S} , see e.g. Proposition I.9.3 in [3]. Thus, the conclusion of Theorem 5.1 follows by Theorem 5 in [27] as well as by Lemma 3.1 and Remark 3.1. \square

6 On extending to the boundary of the direct mappings

As it was before, we assume here that the function K_f is extended by zero outside of the domain D .

In contrast to the case of the inverse mappings, as it was already established in the plane, no degree of integrability of the dilatation leads to the extension to the boundary of direct mappings of the Sobolev class, see e.g. the proof of Proposition 6.3 in [20]. The corresponding criterion for that given below is much more refined. Namely, in view of Lemma 3.1 and Remark 3.1, by Lemma 3 in [27] we obtain the following.

Lemma 6.1 *Let \mathbb{S} and \mathbb{S}^* be Riemann surfaces, D and D^* be domains in $\bar{\mathbb{S}}$ and $\bar{\mathbb{S}}^*$, correspondingly, $\partial D \subset \mathbb{S}$, $\partial D^* \subset \mathbb{S}^*$, D be locally connected at a point $p_0 \in \partial D$. Suppose that $f : D \rightarrow D^*$ is a homeomorphism of finite distortion with $K_f \in L^1_{\text{loc}}$ and ∂D^* is strongly accessible at least at one point in the cluster set $C(p_0, f)$ and*

$$\int_{\varepsilon < h(p, p_0) < \varepsilon_0} K_f(p) \cdot \psi_{p_0, \varepsilon}^2(h(p, p_0)) dh(p) = o(I_{p_0, \varepsilon_0}^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad (6.1)$$

for some $\varepsilon_0 > 0$ where $\psi_{p_0, \varepsilon}(t)$ is a family of nonnegative measurable (by Lebesgue) functions on $(0, \infty)$ such that

$$0 < I_{p_0, \varepsilon_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{p_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0) . \quad (6.2)$$

Then f is extended by continuity to the point p_0 and $f(p_0) \in \partial D^*$.

Note that conditions (6.1)-(6.2) imply that $I_{p_0, \varepsilon_0}(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$ and that ε_0 can be chosen arbitrarily small with keeping (6.1)-(6.2).

Lemma 6.1 makes possible to obtain a series of criteria on the continuous extension to the boundary of mappings with finite distortion between domains on Riemann surfaces. Here we assume that $K_f \equiv 0$ outside of D .

Theorem 6.1 *Let \mathbb{S} and \mathbb{S}^* be Riemann surfaces, D and D^* be domains on $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}^*}$, correspondingly, $\partial D \subset \mathbb{S}$ and $\partial D^* \subset \mathbb{S}^*$, D be locally connected on its boundary and ∂D^* be strongly accessible. Suppose that $f : D \rightarrow D^*$ is a homeomorphism of finite distortion with $K_f \in L^1_{\text{loc}}$ and*

$$\int_0^\delta \frac{dr}{\|K_f\|(p_0, r)} = \infty \quad \forall p_0 \in \partial D \quad (6.3)$$

where

$$\|K_f\|(p_0, r) = \int_{h(p, p_0)=r} K_f(p) ds_h(p) . \quad (6.4)$$

Then the mapping f is extended by continuity to \overline{D} and $f(\partial D) = \partial D^*$.

Proof. Indeed, setting $\psi_{p_0}(t) = 1/\|K_f\|(p_0, t)$ for all $t \in (0, \varepsilon_0)$ under small enough $\varepsilon_0 > 0$ and $\psi_{p_0}(t) = 1$ for all $t \in (\varepsilon_0, \infty)$, we obtain from condition (6.3) that

$$\int_{\varepsilon < h(p, p_0) < \varepsilon_0} K_f(p) \cdot \psi_{p_0}^2(h(p, p_0)) dh(p) = I_{p_0, \varepsilon_0}(\varepsilon) = o(I_{p_0, \varepsilon_0}^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0$$

where, in view of the conditions $K_f(p) \geq 1$ in D and $K_f \in L^1_{\text{loc}}$,

$$0 < I_{p_0, \varepsilon_0}(\varepsilon) := \int_\varepsilon^{\varepsilon_0} \psi_{p_0}(t) dt < \infty .$$

Thus, the first conclusion of Theorem 6.1 follows from Lemma 6.1. The second conclusion of Theorem 6.1 follows e.g. from Proposition 2.5 in [23], see also Proposition 13.5 in [20]. \square

Corollary 6.1 *In particular, the conclusion of Theorem 6.1 holds if*

$$K_f(p) = O\left(\log \frac{1}{h(p, p_0)}\right) \quad \text{as } p \rightarrow p_0 \quad \forall p_0 \in \partial D \quad (6.5)$$

or, more generally,

$$k_{p_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall p_0 \in \partial D \quad (6.6)$$

where $k_{p_0}(\varepsilon)$ is the mean value of the function K_f over the circle $h(p, p_0) = \varepsilon$.

By Theorem 3.1 in [24] with $\lambda_2 = e/\pi$ we have the following consequence from Theorem 6.1, see also arguments in the proof of Lemma 3.1.

Theorem 6.2 *Under hypotheses of Theorem 6.1, suppose that*

$$\int_U \Phi(K_f(p)) \, dh(p) < \infty \quad (6.7)$$

in a neighborhood U of ∂D where $\Phi : \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ is a nondecreasing convex function with the condition

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty, \quad \delta > \Phi(0). \quad (6.8)$$

Then the mapping f is extended by continuity to \overline{D} and $f(\partial D) = \partial D^*$.

Remark 6.1 Note by Theorem 5.1 and Remark 5.1 in [14] condition (6.8) is not only necessary but also sufficient for the continuous extension to the boundary of all mappings f of finite distortion with integral restrictions of the form (6.7). Note also that by Theorem 2.1 in [24] condition (6.8) is equivalent to each of the following conditions where $H(t) = \log \Phi(t)$:

$$\int_{\Delta}^{\infty} H'(t) \frac{dt}{t} = \infty, \quad (6.9)$$

$$\int_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty, \quad (6.10)$$

$$\int_{\Delta}^{\infty} H(t) \frac{dt}{t^2} = \infty \quad (6.11)$$

for some $\Delta > 0$, and also to each of the equality:

$$\int_0^{\delta} H\left(\frac{1}{t}\right) dt = \infty \quad (6.12)$$

for some $\delta > 0$,

$$\int_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \quad (6.13)$$

for some $\Delta_* > H(+0)$.

Here the integral in (6.10) is understood as the Lebesgue-Stieltjes integral, and the integrals in (6.9), (6.11)–(6.13) as the usual Lebesgue integrals.

It is necessary to give more explanations. In the right hand sides of conditions (6.9)–(6.13), we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0, t_*]$, then $H(t) = -\infty$ for $t \in [0, t_*]$, and we complete the definition in (6.9) setting $H'(t) = 0$ for $t \in [0, t_*]$. Note that conditions (6.10) and (6.11) exclude that t_* belongs to the interval of integrability because in the contrary case the left hand sides in (6.10) and (6.11) either are equal $-\infty$ or not determined. Hence we may assume that in (6.9)–(6.12) $\delta > t_0$, correspondingly, $\Delta < 1/t_0$ where $t_0 := \sup_{\Phi(t)=0} t$ and $t_0 = 0$ if $\Phi(0) > 0$.

Among the conditions counted above, the most interesting one is condition (6.11) that can be written in the form:

$$\int_{\delta}^{\infty} \log \Phi(t) \frac{dt}{t^2} = \infty. \quad (6.14)$$

Corollary 6.2 *In particular, the conclusion of Theorem 6.2 holds if, for some $\alpha > 0$,*

$$\int_U e^{\alpha K_f(p)} dh(p) < \infty. \quad (6.15)$$

The following statement follows from Lemma 6.1 for $\psi(t) = 1/t$.

Theorem 6.3 *Under the hypotheses of Theorem 6.1, if*

$$\int_{\varepsilon < h(p, p_0) < \varepsilon_0} K_f(p) \frac{dh(p)}{h(p, p_0)^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall p_0 \in \partial D, \quad (6.16)$$

then the mapping f is extended by continuity to \overline{D} and $f(\partial D) = \partial D^$.*

Remark 6.2 Choosing in Lemma 6.1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we obtain that condition (6.16) can be replaced by the conditions

$$\int_{\varepsilon < h(p, p_0) < \varepsilon_0} \frac{K_f(p) dh(p)}{\left(h(p, p_0) \log \frac{1}{h(p, p_0)}\right)^2} = o\left(\left[\log \log \frac{1}{\varepsilon}\right]^2\right) \quad \text{as } \varepsilon \rightarrow 0. \quad (6.17)$$

Similarly, condition (6.6) by Theorem 6.1 can be replaced by the weaker condition

$$k_{z_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0. \quad (6.18)$$

Of course, we could give here a series of the corresponding conditions of the logarithmic type applying suitable functions $\psi(t)$.

Following paper [23], cf. [9], see also Section 13.4 in [20], Section 2.3 in [7], we say that a function $\varphi : \mathbb{S} \rightarrow \mathbb{R}$ has **finite mean oscillation** at a point $p_0 \in \mathbb{S}$, written $\varphi \in \text{FMO}(p_0)$, if

$$\limsup_{\varepsilon \rightarrow 0} \int_{B(p_0, \varepsilon)} |\varphi(p) - \tilde{\varphi}_\varepsilon| dh(p) < \infty \quad (6.19)$$

where $\tilde{\varphi}_\varepsilon$ is the mean value of φ over the disk $B(p_0, \varepsilon) = \{p \in \mathbb{S} : h(p, p_0) < \varepsilon\}$.

By Remark 3.1 and Lemma 6.1 with the choice $\psi_{p_0, \varepsilon}(t) \equiv 1/t \log \frac{1}{t}$, in view of Lemma 4.1 and Remark 4.1 in [23], see also Lemma 13.2 and Remark 13.3 in [20], we obtain the following result.

Theorem 6.4 *If under the hypotheses of Theorem 6.1, for some $Q : \mathbb{S} \rightarrow \mathbb{R}^+$,*

$$K_f(p) \leq Q(p) \in \text{FMO}(p_0) \quad \forall p_0 \in \partial D. \quad (6.20)$$

Then the mapping f is extended by continuity to \overline{D} and $f(\partial D) = \partial D^$.*

By Corollary 4.1 in [23], see also Corollary 13.3 in [20], we have also from Theorem 6.4 the next statement:

Corollary 6.3 *In particular, the conclusion of Theorem 6.4 holds if*

$$\limsup_{\varepsilon \rightarrow 0} \int_{B(p_0, \varepsilon)} K_f(p) dh(p) < \infty \quad \forall p_0 \in \partial D . \quad (6.21)$$

Remark 6.3 Note that Lemma 6.1 makes possible also to realize the point-wise analysis: if the given conditions for the dilatation hold at one boundary point of D , then the extension of the mappings by continuity holds at this point. However, not to be repeated we will not formulate here the corresponding point-wise results in the explicit form.

7 On homeomorphic extension to the boundary

Combining Theorem 5.1 and results of the last section, we obtain a series of effective criteria of the homeomorphic extension to the boundary of the mappings with finite distortion between domains on Riemann surfaces. As it was before, here we assume that the function K_f is extended by zero outside of the domain D .

Theorem 7.1 *Let under the hypotheses of Theorem 5.1*

$$\int_0^\delta \frac{dr}{||K_f||(p_0, r)} = \infty \quad \forall p_0 \in \partial D \quad (7.1)$$

where

$$||K_f||(p_0, r) = \int_{h(p, p_0)=r} K_f(p) ds_h(p) . \quad (7.2)$$

Then the mapping f is extended to the homeomorphism of \overline{D} onto $\overline{D^}$.*

Corollary 7.1 *In particular, the conclusion of Theorem 7.1 holds if*

$$K_f(p) = O\left(\log \frac{1}{h(p, p_0)}\right) \quad \text{as } p \rightarrow p_0 \quad \forall p_0 \in \partial D \quad (7.3)$$

or, more generally,

$$k_{p_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall p_0 \in \partial D \quad (7.4)$$

where $k_{p_0}(\varepsilon)$ is the mean value of the function K_f over the circle $h(p, p_0) = \varepsilon$.

Theorem 7.2 *Under the hypotheses of Theorem 5.1, suppose that*

$$\int_U \Phi(K_f(p)) dh(p) < \infty \quad (7.5)$$

in a neighborhood U of ∂D where $\Phi : \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ is a nondecreasing convex function with the condition

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \quad (7.6)$$

for some $\delta > \Phi(0)$. Then the mapping f is extended to a homeomorphism of \overline{D} onto $\overline{D^*}$.

Corollary 7.2 *In particular, the conclusion of Theorem 7.2 holds if, for some $\alpha > 0$, in a neighborhood U of ∂D*

$$\int_U e^{\alpha K_f(p)} dh(p) < \infty. \quad (7.7)$$

Theorem 7.3 *Let under the hypotheses of Theorem 5.1*

$$\int_{\varepsilon < h(p, p_0) < \varepsilon_0} K_f(p) \frac{dh(p)}{h(p, p_0)^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall p_0 \in \partial D. \quad (7.8)$$

Then the mapping f is extended to a homeomorphism of \overline{D} onto $\overline{D^*}$.

Theorem 7.4 *Let under the hypotheses of Theorem 5.1, for some $Q : \mathbb{S} \rightarrow \mathbb{R}^+$,*

$$K_f(p) \leq Q(p) \in \text{FMO}(p_0) \quad \forall p_0 \in \partial D. \quad (7.9)$$

Then the mapping f is extended to a homeomorphism of \overline{D} onto $\overline{D^}$.*

Corollary 7.3 *In particular, the conclusion of Theorem 7.4 holds if*

$$\limsup_{\varepsilon \rightarrow 0} \int_{B(p_0, \varepsilon)} K_f(p) dh(p) < \infty \quad \forall p_0 \in \partial D. \quad (7.10)$$

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